

ON THE DYNAMICS OF A TIME-PERIODIC EQUATION

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ABSTRACT. In this paper we use the second order equation

$$\frac{d^2q}{dt^2} + (\lambda - \gamma q^2) \frac{dq}{dt} - q + q^3 = \mu q^2 \sin \omega t$$

as a demonstrative example to illustrate how to apply the analysis of [WO] and [WOk] to the studies of concrete equations. We prove, among many other things, that there are positive measure sets of parameters $(\lambda, \gamma, \mu, \omega)$ corresponding to the case of intersected (See Fig. 1(a)) and the case of separated (See Fig. 1(b)) stable and unstable manifold of the solution $q(t) = 0$, $t \in \mathbb{R}$ respectively, so that the corresponding equations admit strange attractors with SRB measures.

In the history of the theory of dynamical systems, ordinary differential equations have served as a source of inspirations and a test ground. There are mainly two ways in relating the studies of differential equations to the studies of maps, both originated from the work of H. Poincaré. The first is to use return maps locally defined on Poincaré sections for the studies of autonomous equations. The second is to use the globally defined time-T maps for the studies of time-periodic equations.

The studies of periodically forced second order equations, such as van de Pol's equation, Duffing's equation and the equation for non-linear pendulums, have played substantial roles in shaping the chaos theory in modern times ([Ar], [D], [V], [Lev1], [Le], [GH]). When a homoclinic solution is periodically perturbed, the stable and the unstable manifold of the perturbed saddle intersect each other within a certain range of forcing parameters, generating homoclinic tangles and chaotic dynamics ([P], [S], [M]). See Fig. 1(a). There are also forcing parameters for which the stable and the unstable manifold of the perturbed saddle are pulled apart. See Fig. 1(b).

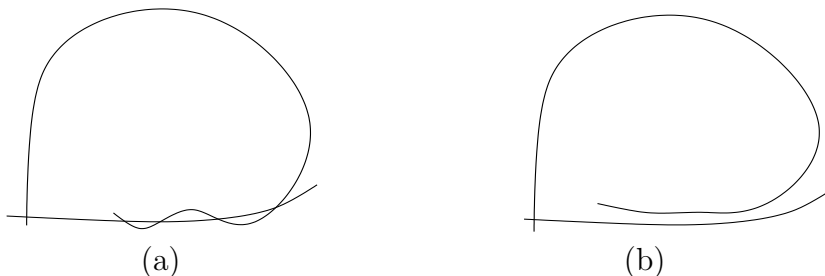


Fig. 1 The stable and the unstable manifold of a perturbed saddle.

With respect to the two scenarios of Fig. 1, there has been a (somewhat incorrect) perception that the scenario depicted in Fig. 1(a) is much more interesting than that in Fig. 1(b), mainly due to the fact that Fig. 1(a) induces complicated dynamics while Fig 1(b) appears simple. A computational procedure (the Melnikov's method) has been developed to distinguish the two scenarios. See Sects. 4.5 and 4.6 of [GH] for an exposition on Melnikov's method and its applications to a list of periodically perturbed equations. Historically, homoclinic tangles of Fig. 1(a) have been a strong influence, yet until very recently in the analysis of concrete equations, little has been rigorously proved beyond the existence of embedded horseshoes.

In [WO] and [WOk], we have connected the studies of periodically perturbed equations in the vicinity of a dissipative homoclinic solution to the studies of two specific class of maps. The main idea, motivated by [AS], is to abandon the globally defined time-T maps, and in its stead to compute the return maps in the extended phase space around the unperturbed homoclinic loop. It has turned out that (1) the dynamics of homoclinic tangles of Fig. 1(a) are those of a family of infinitely wrapped horseshoe maps [WOk] (see Sect. 1.2 for details), and (2) the dynamics of Fig. 1(b) are those of a family of rank one maps [WO]. With the return maps obtained in [WO] and in [WOk], we are now able to apply many existing theories on maps to the studies of periodically perturbed equations with dissipative homoclinic saddle. Among the theories directly applied are the Newhouse theory [N], [PT] on homoclinic tangency; the theory of SRB measures [Si], [R], [Bo]; the theory of Hénon-like attractors [BC2], [MV], [BY]; and the recent theory of rank one chaos [WY1]-[WY3] based on the theory of Benedicks and Carleson on strongly dissipative Hénon maps [BC2]. This method has led us to many new results.

In this paper we illustrate how to apply the analysis of [WO] and [WOk] to the studies of concrete equations. We use the equation

$$(0.1) \quad \frac{d^2 q}{dt^2} + (\lambda - \gamma q^2) \frac{dq}{dt} - q + q^3 = \mu q^2 \sin \omega t$$

as a demonstrative example to prove, among many other things, that there exist positive measure sets of parameters $(\lambda, \gamma, \mu, \omega)$ corresponding respectively to the dynamics scenarios of Fig. 1(a) and Fig. 1(b), so that equation (0.1) admits strange attractors with SRB measures. We note that the unperturbed part of equation (0.1) has been studied previously in [HR].

1. STATEMENT OF RESULTS

Let us re-write equation (0.1) as

$$(1.1) \quad \begin{aligned} \frac{dq}{dt} &= p \\ \frac{dp}{dt} &= -(\lambda - \gamma q^2)p + q - q^3 + \mu q^2 \sin \theta \\ \frac{d\theta}{dt} &= \omega. \end{aligned}$$

The space of (q, p, θ) is the extended phase space for equation (0.1) where $\theta \in S^1$.

1.1. Return maps in extended phase space. We call a saddle fixed point of a two dimensional autonomous system a *dissipative* saddle if the magnitude of the associated negative eigenvalue is larger than that of the positive eigenvalue. We call a saddle point a *homoclinic* saddle if it is approached by one solution, which we call a *homoclinic* solution, from both the positive and the negative directions of time. The autonomous part of equation (0.1) is written as

$$(1.2) \quad \frac{d^2 q}{dt^2} + (\lambda - \gamma q^2) \frac{dq}{dt} - q + q^3 = 0,$$

where $\lambda > 0$ and γ are parameters. Let $(q, p) \in \mathbb{R}^2$ be the phase space where $p = \frac{dq}{dt}$. We start with a previously existed result on (1.2) [HR].

Proposition 1.1 (Dissipative homoclinic saddle). *There exists $\lambda_0 > 0$ sufficiently small, such that for $\lambda \in [0, \lambda_0)$, there exists a $\gamma_\lambda, |\gamma_\lambda| < 10\lambda$ such that for $\gamma = \gamma_\lambda$;*

(i) *equation (1.2) has a homoclinic solution for $(q, p) = (0, 0)$, which we denote as*

$$\ell_\lambda = \{\ell_\lambda(t) = (a_\lambda(t), b_\lambda(t)), t \in \mathbb{R}\},$$

satisfying $a_\lambda(t) > 0$ for all t ;

(ii) *for any given $L > 0$, there exists a $K(L)$ independent of λ , such that for all $t \in [-L, L]$,*

$$|\ell_\lambda(t) - \ell_0(t)| < K(L)\lambda.$$

See also [W] for an alternative proof of this proposition. Note that $\ell_0(t)$ is a homoclinic solution of equation (1.2) for $\lambda = \gamma = 0$. Let

$$(1.3) \quad \alpha = \frac{1}{2}(\sqrt{\lambda^2 + 4} + \lambda), \quad \beta = \frac{1}{2}(\sqrt{\lambda^2 + 4} - \lambda).$$

Then $-\alpha, \beta$ are the two eigenvalues of $(0, 0)$. Since $-\alpha + \beta = -\lambda < 0$, $(0, 0)$ is a dissipative saddle provided that $\lambda > 0$.

Let $\lambda_0 > 0$ be sufficiently small. We regard the value of λ_0 as been fixed. We say that $\lambda \in (0, \lambda_0)$ is *non-resonant* if α, β satisfy certain Diophantine non-resonance condition. Namely,

(H) *there exists $d_1, d_2 > 0$ so that for all $n, m \in \mathbb{Z}^+$,*

$$|n\alpha - m\beta| > d_1(|n| + |m|)^{-d_2}.$$

From this point on we fixed the value of $\lambda \in (0, \lambda_0)$ and assume that it satisfies (H). Let γ_λ be as in Proposition 1.1. We introduce a new parameter ρ for γ by letting

$$(1.4) \quad \gamma = \gamma_\lambda + \mu\rho.$$

ω, ρ, μ are now the parameters of equation (0.1).

We turn now to equation (1.1). Let $\ell = \ell_\lambda$ be the homoclinic loop of Proposition 1.1 in the space of (q, p) . We construct a small neighborhood of ℓ by taking the union

of a small neighborhood U_ε of $(0,0)$ and a small neighborhood D around ℓ out of $U_{\frac{1}{4}\varepsilon}$. See Fig. 2. Let $\sigma^\pm \in U_\varepsilon \cap D$ be the two line segments depicted in Fig. 2, both perpendicular to the homoclinic solution. In the space of (q, p, θ) we denote

$$\mathcal{U}_\varepsilon = U_\varepsilon \times S^1, \quad \mathbf{D} = D \times S^1$$

and let

$$\Sigma^\pm = \sigma^\pm \times S^1.$$

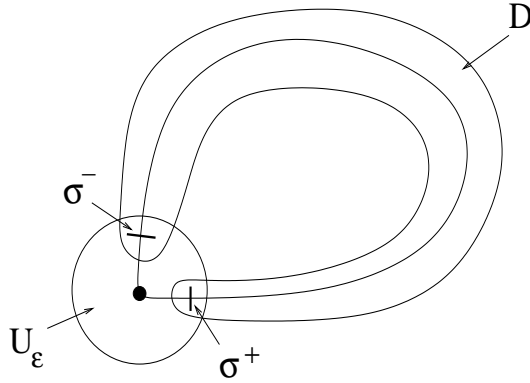


Fig. 2 U_ε , D and σ^\pm .

Let $\mathcal{N} : \Sigma^+ \rightarrow \Sigma^-$ be the maps induced by the solutions of (1.1) on \mathcal{U}_ε and $\mathcal{M} : \Sigma^- \rightarrow \Sigma^+$ be the maps induced by the solutions of (1.1) on \mathbf{D} . See Fig. 3. Let $\mathcal{F}_{\omega, \rho, \mu} := \mathcal{N} \circ \mathcal{M} : \Sigma^- \rightarrow \Sigma^-$ be the return map defined by equation (1.1). We compute $\mathcal{F} = \mathcal{F}_{\omega, \rho, \mu}$ following the steps of [WO] and [Wok].

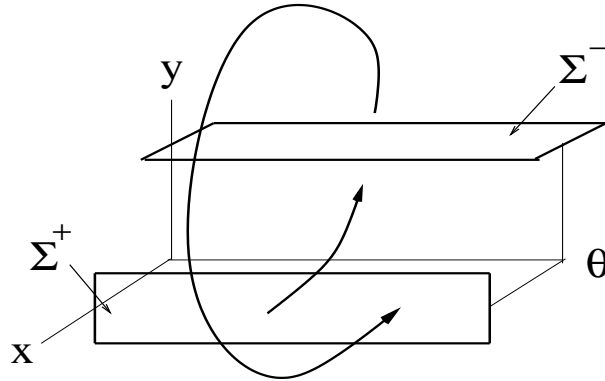


Fig. 3 \mathcal{N} and \mathcal{M} .

We now define the range of parameters for this paper. Let R_ω be sufficiently large, and R_ω, R_ρ and R_μ be such that

$$R_\omega \ll R_\rho \ll R_\mu.$$

We only consider parameters (ω, ρ, μ) inside of

$$\mathbb{P} = \{(\omega, \rho, \mu) : \omega \in (0, R_\omega), \rho \in (0, R_\rho), \mu \in (0, R_\mu^{-1})\}.$$

In the extended phase space (q, p, θ) , $q = p = 0$ is a periodic orbit. We study equation (1.1), denoting the stable and the unstable manifold of the solution $(q, p) = (0, 0)$ as W^s and W^u respectively. First we have

Theorem 1 (Surface of tangency). *There exists a continuous function $S^*(\omega, \mu) : (0, R_\omega) \times (0, R_\mu^{-1}) \rightarrow (0, R_\rho)$ such that*

- (a) $W^u \cap W^s \neq \emptyset$ if $0 < \rho < S_\mu^*(\omega, \mu)$;
- (b) $W^u \cap W^s = \emptyset$ if $S_\mu^*(\omega, \mu) < \rho < R_\rho$.

So under the surface S^* defined by $\rho = S^*(\omega, \mu)$ is the scenario of homoclinic tangles of Fig. 1(a) and above this surface is the scenario of Fig. 1(b). S^* is one of the surfaces (the lowest one of the three) depicted in Fig. 6 in Sect. 1.3.

1.2. Homoclinic tangles. Assume that $(\omega, \rho, \mu) \in \mathbb{P}$ is beneath the surface S^* of Theorem 1. We focus on the set of solutions that stay inside of $\mathcal{U}_\varepsilon \cup \mathbf{D}$ for all time. This set of solutions is the homoclinic tangle in the vicinity of ℓ_λ . If $(\omega, \rho, \mu) \in \mathbb{P}$ is beneath S^* , then the return map $\mathcal{F} = \mathcal{F}_{\omega, \rho, \mu}$ is only partially defined on Σ^- because some of the images of Σ^- under \mathcal{M} would first hit on the wrong side of the local stable manifold of $(q, p) = (0, 0)$ in Σ^+ , then sneak out of \mathcal{U}_ε . See Fig. 4.

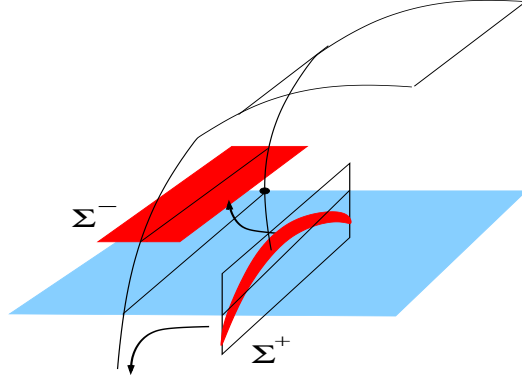


Fig. 4 Partial returns to Σ^- .

We prove that \mathcal{F} is an infinitely wrapped horseshoe map, the geometric structure of which is as follows. Take an annulus $\mathcal{A} = S^1 \times I$ (This is Σ^- for \mathcal{F}). We represent points in S^1 and I by using variables θ and z respectively. We call the direction of θ the horizontal direction and the direction of z the vertical direction. To form an infinitely wrapped horseshoe map, which we denote as \mathcal{F} , we first divide \mathcal{A} into two vertical strips, denoted as V and U . $\mathcal{F} : V \rightarrow \mathcal{A}$ is defined on V but not on U . We compress V in the vertical direction and stretch it in the horizontal direction, making the image infinitely long towards both ends. Then we fold it and wrap it around the annulus \mathcal{A} infinitely many times. See Fig. 5.

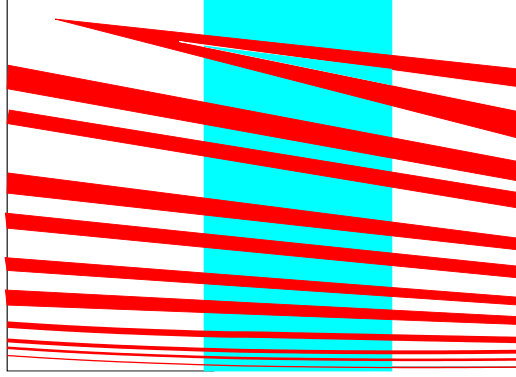


Fig. 5 Infinitely wrapped horseshoe maps.

Let

$$\Omega_{\mathcal{F}} = \{(\theta, z) \in V : \mathcal{F}^n(\theta, z) \in V \quad \forall n \geq 0\}, \quad \Lambda_{\mathcal{F}} = \bigcap_{n \geq 0} \mathcal{F}^n(\Omega_{\mathcal{F}}).$$

The dynamic structure of the homoclinic tangle in the vicinity of ℓ_λ is manifested in that of $\Lambda_{\mathcal{F}}$. $\Lambda_{\mathcal{F}}$ obviously contains a horseshoe of infinitely many symbols. This horseshoe covers Smale's horseshoe and all its variations. It is the one that resides inside all homoclinic tangles.

The structure of $\Omega_{\mathcal{F}}$ and $\Lambda_{\mathcal{F}}$ depends sensitively on the location of the folded part of $\mathcal{F}(V)$. If this part is deep inside of U , then the entire homoclinic tangle is reduced to one horseshoe of infinitely many symbols. If it is located inside of V , then the homoclinic tangles are likely to have attracting periodic solutions, or sinks and observable chaos associated with non-degenerate transversal homoclinic tangency. In particular, we have

Theorem 2 (Dynamics of homoclinic tangles). *There exists an open domain in the plane of (ω, ρ) , which we denote as $DH \subset (0, R_\omega) \times (0, R_\rho)$, such that $DH \times (0, R_\mu^{-1})$ is strictly under the surface S^* in \mathbb{P} ; and for any fixed $(\omega, \rho) \in DH$, the return maps $\mathcal{F}_\mu = \mathcal{F}_{\omega, \rho, \mu}$, as a one parameter family in μ , satisfy the follows:*

(i) *There exists a sequence of μ , accumulating at $\mu = 0$, which we denote as*

$$R_\mu^{-1} \geq \mu_1^{(r)} > \mu_1^{(l)} > \cdots > \mu_n^{(r)} > \mu_n^{(l)} > \cdots > 0$$

such that for all $\mu \in [\mu_n^{(l)}, \mu_n^{(r)}]$, $\mathcal{F} = \mathcal{F}_\mu : \Lambda_{\mathcal{F}} \rightarrow \Lambda_{\mathcal{F}}$ conjugates to a full shift of countably many symbols.

(ii) *Inside each of the intervals $[\mu_{n+1}^{(r)}, \mu_n^{(l)}]$, $n > 0$, there exists μ so that \mathcal{F}_μ admits stable periodic solutions.*

(iii) *There are also parameters in each of the intervals $[\mu_{n+1}^{(r)}, \mu_n^{(l)}]$, $n > 0$, such that \mathcal{F}_μ admits non-degenerate transversal homoclinic tangency of a dissipative saddle fixed point.*

Remarks: (1) All results stated so far in this subsection about homoclinic tangles are new. We are not aware of any previous results of the same kind in the analysis of any concrete time-periodic second order equations.

(2) Theorem 2(i) is much stronger than the mere existence of an embedded horseshoe. For these parameters, the entire homoclinic tangle around ℓ_λ contains not only nothing less [S], but also nothing more, than a uniformly hyperbolic invariant set conjugating to a horseshoe of countably many symbols. Since the attractive basin of such horseshoe is of Lebesgue measure zero, these tangles are not observable if one adopt the probabilistic point of view that only events of positive Lebesgue measures are observable in phase space.

(3) Sinks of Theorem 2(ii) are sinks of very strong contraction. They are not related to those obtained through Newhouse theory. The existence of Newhouse sinks follows from Theorem 2(iii).

(4) It also follows from combining Theorem 2(iii) with the result of [MV] that there are positive measure set of parameters μ , such that the return maps \mathcal{F} admit Hénon-like attractors. In contrast to the horseshoes of Theorem 2(i), these are chaos in observable forms (SRB measures) [MV], [BY].

1.3. Invariant curves and rank one chaos. Assume that $(\omega, \rho, \mu) \in \mathbb{P}$ is above the surface S^* of Theorem 1. We are in the case of Fig. 1(b). The return maps \mathcal{F} are now well-defined on Σ^- , and they are not in anyway less interesting than those for homoclinic tangles of Section 1.2 in terms of chaos dynamics. For these parameters, not only the return maps are rank one maps, to which the theory of [WY1] and [WY2] directly apply, but also they fall naturally into the specific category of rank one maps previously studied by Wang and Young in [WY4]. For these parameters the dynamics of equation (1.1) around ℓ_λ are determined by the magnitude of the forcing frequency ω . When the forcing frequency ω is small, we obtain, for all μ sufficiently small, an attracting tori in the extended phase space. In particular, we obtain an attracting tori consisting of quasi-periodic solutions for a set of μ with positive Lebesgue density at $\mu = 0$. As ω increases, the attracting tori is dis-integrated into isolated periodic sinks and saddles. Increasing the forcing frequency ω further, the stable and the unstable manifold of these periodic saddles will fold, and intersect to create horseshoe and strange attractors. We have, in particular, that these are strange attractors with SRB measures. In particular, we have

Theorem 3 (Invariant curves and quasi-periodic tori). *There exists a $\rho_0 > 0$ and a continuous function $Q : (\rho_0, R_\rho) \times (0, R_\mu^{-1}) \rightarrow (0, R_\omega)$, such that for any given $(\omega, \rho, \mu) \in \mathbb{P}$ that is on the left hand side of the surface Q defined by $\omega = Q(\rho, \mu)$, that is, these satisfying $\omega < Q(\rho, \mu)$, we have the follows:*

(1) *The return map $\mathcal{F} : \Sigma^- \rightarrow \Sigma^-$ admits a globally attracting, simply closed invariant curve; and the maps induced on this invariant curve is equivalent to a circle diffeomorphism.*

(2) *There is an open set $DI \subset (0, R_\omega) \times (\rho_0, R_\rho)$ in the (ω, ρ) -plane, such that $DI \times (0, R_\mu^{-1})$ is located on the left hand side of the surface Q . Let $(\omega, \rho) \in DI$ be fixed and regard $\mathcal{F}_\mu = \mathcal{F}_{\omega, \rho, \mu}$ as a one parameter family. Then there exists a set of μ of positive Lebesgue density at $\mu = 0$, such that the circle diffeomorphism induced on the attracting invariant curve above is with an irrational rotation number.*

Remark: See Fig. 6 for Q inside of \mathbb{P} . Similar results are previously obtained for periodically kicked limit cycles in [WY4], [WY5], [LWY], and for periodically perturbed hyperbolic periodic solutions in [Lev2], [H].

We turn to the case of strange attractors and rank one chaos.

Theorem 4 (Strange attractors and rank one chaos). *Let $S^*(\omega, \mu)$ be as in Theorem 2 and*

$$S(\omega, \mu) = (1 + \omega^{\frac{1}{2}})S^*(\omega, \mu).$$

Let $(\omega, \rho, \mu) \in \mathbb{P}$ be such that

$$S^*(\omega, \mu) < \rho < S(\omega, \mu).$$

Then,

(1) the return map $\mathcal{F} : \Sigma^- \rightarrow \Sigma^-$ admits a global attractor that is strange in the sense that it contains an embedded horseshoe provided that $\omega > 100$; and

(2) there exists an open domain $DC \subset (100, R_\omega) \times (0, R_\rho)$ in the (ω, ρ) -plane, such that $DC \times (0, R_\mu^{-1})$ is located in between S^ and the surface S defined by $\rho = S(\omega, \mu)$. Let $(\omega, \rho) \in DC$ be fixed and regard $\mathcal{F}_\mu = \mathcal{F}_{\omega, \rho, \mu}$ as a one parameter family. Then there exists a set of μ of positive Lebesgue density at $\mu = 0$, such that \mathcal{F}_μ admits a strange attractor with an ergodic SRB measure ν . Furthermore, almost every point of Σ^- is generic with respect to ν .*

Remarks: (1) Various dynamical scenarios of Theorems 1-4 are put together in Fig. 6. This figure is self-explanatory. Results similar to Theorem 4 are previously obtained for periodically kicked limit cycles in [WY4], [WY5], [LWY], and for certain slow-fast systems in [GWY].

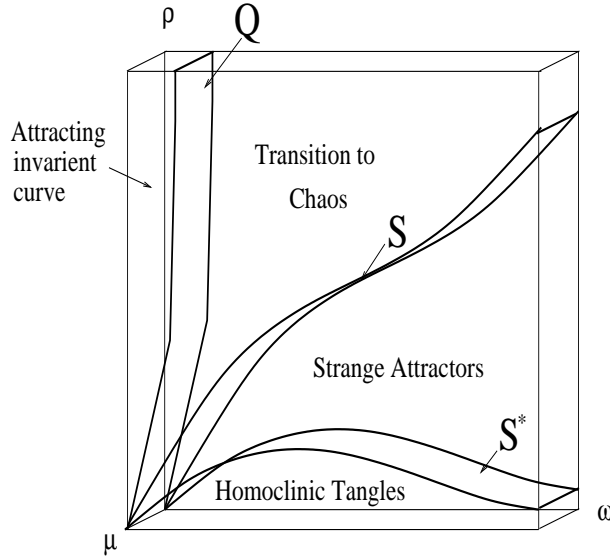


Fig. 6 Dynamical scenarios in parameter space.

(2) Theorems 3 and 4 remain valid if we change the forcing function $\mu q^2 \sin \omega t$ in (0.1) to simply $\mu \sin \omega t$. This case is covered by [WO].

(3) SRB measures are constructed initially for uniformly hyperbolic systems by Sinai, Ruelle and Bowen ([Si], [R], [Bo]). They are observable objects representing statistical order in chaos. We refer to [Y] for a review on the theory of SRB measures. [WY1] contains not only the existence of SRB measures but also a comprehensive dynamical profile for the good maps, including geometric structures of the attractors and statistical properties. We have opted to limit our statement to SRB measures, but all aspects of that larger dynamical picture in fact apply.

(4) Finally we note that the open domains DH , DI and DC in Theorems 2-4 are not in any sense small in size. One main restriction is that their respective product to $(0, R_\mu^{-1})$ do not intersect Q , S and S^* . This is essentially the only restriction for DI of Theorem 3. For DH and DC we also require that ω is reasonably large.

2. DERIVATION OF THE RETURN MAPS

In this section we derive the return maps $\mathcal{F} = \mathcal{N} \circ \mathcal{M}$ of Sect. 1.1 for equation (1.1) assuming Proposition 1.1. In Sect. 2.1 we introduce coordinate changes to transform the equations into canonical forms, which we will use in Sect. 2.2 to compute the return maps.

2.1. Equation in canonical forms. In Sect. 2.1A we normalize the linear part. In Sect. 2.1B we linearize equation (1.1) on \mathcal{U}_ε , the ε -neighborhood of the saddle point. In Sect. 2.1C we introduce a set of new variables and derive the corresponding equations on \mathbf{D} , a small neighborhood around the homoclinic solution ℓ_λ out of $\mathcal{U}_{\frac{1}{4}\varepsilon}$.

A. Normalizing the linear part Let $\lambda \in (0, \lambda_0)$ be fixed satisfying (H) in Sect. 1.1, and $\gamma = \gamma_\lambda + \mu\rho$ where γ_λ is as in Proposition 1.1. We re-write equation (1.1) as

$$(2.1) \quad \begin{aligned} \frac{dq}{dt} &= p \\ \frac{dp}{dt} &= -(\lambda - \gamma_\lambda q^2)p + q - q^3 + \mu(\rho q^2 p + q^2 \sin \theta) \\ \frac{d\theta}{dt} &= \omega. \end{aligned}$$

To put the linear part of equation (2.1) in canonical form, we introduce new variables (x, y) so that

$$(2.2) \quad q = x + \alpha y, \quad p = -\alpha x + y,$$

where $\alpha = \frac{1}{2}(\lambda + \sqrt{\lambda^2 + 4})$ is as in (1.3) in Sect. 1.1. In reverse we have

$$(2.3) \quad x = \frac{1}{1 + \alpha^2}(q - \alpha p) \quad y = \frac{1}{1 + \alpha^2}(\alpha q + p).$$

The new equations for (x, y) are

$$(2.4) \quad \begin{aligned} \frac{dx}{dt} &= -\alpha x + f(x, y) + \mu(A(x, y)\rho + C(x, y) \sin \theta) \\ \frac{dy}{dt} &= \beta y + g(x, y) + \mu(B(x, y)\rho + D(x, y) \sin \theta) \\ \frac{d\theta}{dt} &= \omega \end{aligned}$$

where $\beta = \alpha^{-1}$ is again as in (1.3), and

$$\begin{aligned} f(x, y) &= \frac{\alpha}{1 + \alpha^2} (\gamma_\lambda(x + \alpha y)^2(y - \alpha x) + (x + \alpha y)^3), \\ g(x, y) &= \frac{-1}{1 + \alpha^2} (\gamma_\lambda(x + \alpha y)^2(y - \alpha x) + (x + \alpha y)^3); \\ A(x, y) &= \frac{\alpha}{1 + \alpha^2}(x + \alpha y)^2(y - \alpha x), \quad B(x, y) = \frac{-1}{1 + \alpha^2}(x + \alpha y)^2(y - \alpha x); \\ C(x, y) &= \frac{\alpha}{1 + \alpha^2}(x + \alpha y)^2, \quad D(x, y) = \frac{-1}{1 + \alpha^2}(x + \alpha y)^2. \end{aligned}$$

Observe that the functions f, g, A, B, C, D are all functions of x, y of order at least two. In comparison to the equations studied in [WOk], equation (2.4) is in a slightly different form. In the rest of this section we present the derivations of the return maps following mainly [WOk]. To avoid repetitive writings, we will refer the reader to the corresponding part of [WOk] for the details of a few technical proofs along the way.

Notations (ω, ρ, μ) are the three parameters of equation (1.1). In what follows we replace μ by $\ln \mu$. $\mu \in (0, R_\mu^{-1})$ corresponds to $\ln \mu \in (-\infty, -\ln R_\mu)$. Denote $p = (\omega, \rho, \ln \mu)$, $p \in \mathbb{P} = (0, R_\omega) \times (0, R_\rho) \times (-\infty, -\ln R_\mu)$. ε is not a parameter of equation (1.1) but an auxiliary parameter representing the size of U_ε . In what follows ε is a positive number sufficiently small satisfying

$$(2.5) \quad R_\mu \gg \varepsilon^{-1} \gg R_\rho \gg R_\omega.$$

The letter K is reserved as a generic constant, the value of which varies from line to line. K is allowed to depend on parameters ω, ρ and ε but not μ . We also use $K(\varepsilon)$ to make explicit when a constant is a dependent of ε . Letter K standing alone represents a constant independent of both ε and μ .

The intended formula for the return maps would contain explicit terms and “error” terms, and we aim on C^3 -control on all error terms in this paper. To facilitate our presentation we adopt a specific conventions for indicating controls on magnitude. For a given constant, we write $\mathcal{O}(1)$, $\mathcal{O}(\varepsilon)$ or $\mathcal{O}(\mu)$ to indicate that the magnitude of the constant is bounded by K , $K\varepsilon$ or $K(\varepsilon)\mu$, respectively. For a function of a set V of variables and parameters on a specific domain, we write $\mathcal{O}_V(1)$, $\mathcal{O}_V(\varepsilon)$ or $\mathcal{O}_V(\mu)$ to indicate that the C^r -norm of the function on the specified domain is bounded by K , $K\varepsilon$ or $K(\varepsilon)\mu$, respectively. We chose to specify the domain in the surrounding text rather than explicitly involving it in the notation. For example, $\mathcal{O}_{\theta, p}(\mu)$ represents a

function of θ , the C^3 -norm¹ of which with respect to θ, ω, ρ and $\ln \mu$ is bounded above by $K(\varepsilon)\mu$.

B. Linearization on \mathcal{U}_ε . Let X, Y be such that

$$(2.6) \quad \begin{aligned} x &= X + P(X, Y) + \mu \tilde{P}(X, Y, \theta; p) \\ y &= Y + Q(X, Y) + \mu \tilde{Q}(X, Y, \theta; p) \end{aligned}$$

where $P, Q, \tilde{P}, \tilde{Q}$ as functions of X and Y are real-analytic on $|(X, Y)| < 2\varepsilon$, and the values of these functions and their first derivatives with respect to X and Y at $(X, Y) = (0, 0)$ are all zero. As is explicitly indicated in (2.6), P and Q are independent of θ and p . We also assume that

$$\tilde{P}(X, Y, \theta + 2\pi; p) = \tilde{P}(X, Y, \theta; p), \quad \tilde{Q}(X, Y, \theta + 2\pi; p) = \tilde{Q}(X, Y, \theta; p)$$

are periodic of period 2π in θ and they are also real-analytic with respect to θ and p for all $\theta \in \mathbb{R}$ and $p \in \mathbb{P}$. We have

Proposition 2.1. *There exists a small neighborhood U_ε of $(0, 0)$ in (X, Y) -space, the size of which are completely determined by equation (1.1) and d_1, d_2 in (H), such that there exists an analytic coordinate transformation in the form of (2.6) on $\mathcal{U}_\varepsilon = U_\varepsilon \times S^1$ that transforms equation (2.4) into*

$$\frac{dX}{dt} = -\alpha X, \quad \frac{dY}{dt} = \beta Y, \quad \frac{d\theta}{dt} = \omega.$$

Moreover, the C^4 -norms of $P, Q, \tilde{P}, \tilde{Q}$ as functions of X, Y, θ, μ are all uniformly bounded from above by a constant K that is independent of both ε and μ on $(X, Y) \in U_\varepsilon, \theta \in \mathbb{R}$ and $p \in \mathbb{P}$.

Proof: This is a standard linearization result. For this proposition to hold we need the non-resonant assumption (H) and the fact that f, g, A, B, C, D are terms of order higher than one at $(x, y) = (0, 0)$. See for instance [CLS] for a proof. \square

C. A canonical form around homoclinic loop. We derive a standard form for equation (2.4) around the homoclinic loop of equation (1.2) outside of $\mathcal{U}_{\frac{1}{4}\varepsilon}$. Let $\ell = \ell_\lambda$ be the homoclinic solution of Proposition 1.1. In (x, y) -space we write ℓ as

$$x = a(t), \quad y = b(t)$$

where $t \in \mathbb{R}$ is the time. Let

$$(u(t), v(t)) = \left| \frac{d}{dt} \ell(t) \right|^{-1} \frac{d}{dt} \ell(t)$$

be the unit tangent vector of ℓ at $\ell(t)$. We replace t by s to write this homoclinic loop as $\ell(s) = (a(s), b(s))$. Let

$$\mathbf{e}(s) = (v(s), -u(s)).$$

¹We emphasize that $\ln \mu$, not μ , is regarded as one of the forcing parameters. Derivatives with respect to μ and to $\ln \mu$ differ by a factor μ .

We now introduce new variables (s, z) through

$$(x, y) = \ell(s) + z\mathbf{e}(s).$$

This is to say that

$$(2.7) \quad x = x(s, z) := a(s) + v(s)z, \quad y = y(s, z) := b(s) - u(s)z.$$

Let us further re-scale z by letting $Z = \mu^{-1}z$. (s, Z, θ) are the variables we use in this paragraph.

Before deriving the equations for (s, Z, θ) , we first define the domain on which these equations are derived. Let $-L^-, L^+$ be the times $\ell = \ell_\lambda$ hits $U_{\frac{1}{2}\varepsilon}$. L^\pm are completely determined by ℓ_λ and ε . Defined

$$\mathbf{D} = \{(s, Z, \theta) : s \in [-2L^-, 2L^+], |Z| \leq K_1(\varepsilon), \theta \in \mathbb{P}\}.$$

for some $K_1(\varepsilon)$ independent of μ . We have

Proposition 2.2. *The equation for (s, Z, θ) on \mathbf{D} is in the form of*

$$(2.8) \quad \begin{aligned} \frac{dZ}{dt} &= E(s)Z + (\mathbb{H}_1(s)\rho + \mathbb{H}_2(s)\sin\theta) + \mathcal{O}_{s,Z,\theta;p}(\mu) \\ \frac{ds}{dt} &= 1 + \mathcal{O}_{s,Z,\theta;p}(\mu) \\ \frac{d\theta}{dt} &= \omega \end{aligned}$$

where

$$(2.9) \quad \begin{aligned} E(s) &= v^2(s)(-\alpha + \partial_x f(a(s), b(s))) + u^2(s)(\beta + \partial_y g(a(s), b(s))) \\ &\quad - u(s)v(s)(\partial_y f(a(s), b(s)) + \partial_x g(a(s), b(s))), \end{aligned}$$

and

$$(2.10) \quad \begin{aligned} \mathbb{H}_1(s) &= v(s)A(a(s), b(s)) - u(s)B(a(s), b(s)); \\ \mathbb{H}_2(s) &= v(s)C(a(s), b(s)) - u(s)D(a(s), b(s)). \end{aligned}$$

Proof: The derivations of equation (2.8) is identical to that of equation (2.13) in [WOk]. We note that $K_1(\varepsilon)$ for \mathbf{D} is also defined explicitly in [WOk]. \square

2.2. Derivation of Return Maps. In this subsection we derive the return map $\mathcal{F} = \mathcal{N} \circ \mathcal{M}$ by using Propositions 2.1 and 2.2. In Sect. 2.2A we defined precisely the surfaces Σ^\pm , and discuss the issue of coordinate conversions on these surfaces. In Sect. 2.2B we compute \mathcal{N} by using Proposition 2.1 and in Sect. 2.2C we compute \mathcal{M} by using Proposition 2.2. $\mathcal{F} = \mathcal{N} \circ \mathcal{M}$ is then obtained in Sect. 2.2D.

A. Poincaré sections Σ^\pm . Let (X, Y, θ) be the phase variables of Proposition 2.1. We define Σ^\pm inside of $\mathcal{U}_\varepsilon \cap \mathbf{D}$ by letting

$$\Sigma^- = \{(X, Y, \theta) : Y = \varepsilon, |X| < \mu, \theta \in S^1\},$$

and

$$\Sigma^+ = \{(X, Y, \theta) : X = \varepsilon, |Y| < K_1(\varepsilon)\mu, \theta \in S^1\}.$$

$K_1(\varepsilon)$ is the same as in Proposition 2.2.

Let $q \in \Sigma^+$ or Σ^- . We represent q by using the (X, Y, θ) -coordinates, for which we have $X = \varepsilon$ on Σ^+ and $Y = \varepsilon$ on Σ^- . We can also use (s, Z, θ) of Proposition 2.2 to represent the same q . Before computing the return maps, we need to first attend two issues that are technical in nature. First, we need to derive the defining equations of Σ^\pm for (s, Z, θ) . Second, we need to be able to change variables back and forth from (X, Y, θ) to (s, Z, θ) on Σ^\pm . Let

$$(2.11) \quad \mathbb{X} = \mu^{-1}X, \quad \mathbb{Y} = \mu^{-1}Y.$$

Proposition 2.3. (a) On Σ^+ , we have

- (i) $s = -L^- + \mathcal{O}_{Z,\theta,p}(\mu)$; and
- (ii) $\mathbb{Y} = (1 + \mathcal{O}(\varepsilon))Z + \mathcal{O}_{\theta,p}(1) + \mathcal{O}_{Z,\theta,p}(\mu)$.

(b) On Σ^- , we have

- (i) $s = -L^- + \mathcal{O}_{Z,\theta,p}(\mu)$; and
- (ii) $Z = (1 + \mathcal{O}(\varepsilon))\mathbb{X} + \mathcal{O}_{\theta,p}(1) + \mathcal{O}_{\mathbb{X},\theta,p}(\mu)$.

Proof: The proof of (a) is identical to that of Lemma 3.1 and 3.3 of [WOk] and the proof of (b) is identical to that of Lemma 3.4 of [WOk]. \square

B. The induced map $\mathcal{N} : \Sigma^+ \rightarrow \Sigma^-$. For $(\mathbb{X}, \mathbb{Y}, \theta) \in \Sigma^+$ we have $\mathbb{X} = \varepsilon\mu^{-1}$ by definition. Similarly, for $(\mathbb{X}, \mathbb{Y}, \theta) \in \Sigma^-$ we have $\mathbb{Y} = \varepsilon\mu^{-1}$. Denote a point on Σ^+ by using (\mathbb{Y}, θ) and a point on Σ^- by using (\mathbb{X}, θ) , and let

$$(\mathbb{X}_1, \theta_1) = \mathcal{N}(\mathbb{Y}, \theta)$$

for $(\mathbb{Y}, \theta) \in \Sigma^+$.

Proposition 2.4. We have for $(\mathbb{Y}, \theta) \in \Sigma^+$,

$$(2.12) \quad \begin{aligned} \mathbb{X}_1 &= (\mu\varepsilon^{-1})^{\frac{\alpha}{\beta}-1} \mathbb{Y}^{\frac{\alpha}{\beta}} \\ \theta_1 &= \theta + \frac{\omega}{\beta} \ln(\varepsilon\mu^{-1}) - \frac{\omega}{\beta} \ln \mathbb{Y}. \end{aligned}$$

Proof: Let T be the time it takes for the solution of the linearized equation of Proposition 2.1 from $(\varepsilon, Y, \theta) \in \Sigma^+$ to get to $(X_1, \varepsilon, \theta_1) \in \Sigma^-$. We have

$$X_1 = \varepsilon e^{-\alpha T}, \quad \varepsilon = Y e^{\beta T}, \quad \theta_1 = \theta + \omega T,$$

from which (2.12) follows. \square

C. The induced map $\mathcal{M} : \Sigma^- \rightarrow \Sigma^+$. Let $\mathbb{H}_i(s)$, $i = 1, 2$ be as in (2.10). In what follows, we write

$$(2.13) \quad \begin{aligned} A_L &= \int_{-L^-}^{L^+} \mathbb{H}_1(s) e^{-\int_0^s E(\tau) d\tau} ds \\ \phi_L(\theta) &= \int_{-L^-}^{L^+} \mathbb{H}_2(s) \sin(\theta + \omega s + \omega L^-) e^{-\int_0^s E(\tau) d\tau} ds. \end{aligned}$$

We also write

$$(2.14) \quad P_L = e^{\int_{-L^-}^{L^+} E(s) ds}, \quad P_L^+ = e^{\int_0^{L^+} E(s) ds}.$$

Note that for P_L we integrate from $s = -L^-$ to $s = L^+$, while for P_L^+ the integration starts from $s = 0$. First we have

Lemma 2.1.

$$P_L \sim \varepsilon^{\frac{\alpha}{\beta} - \frac{\beta}{\alpha}} \ll 1, \quad P_L^+ \sim \varepsilon^{-\frac{\beta}{\alpha}} \gg 1.$$

Proof: Same as Lemma 3.5 in [WOk]. \square

For $q = (s^-, Z, \theta) \in \Sigma^-$, the value of s^- is uniquely determined by that of (Z, θ) through Proposition 2.4(b)(i). So we can use (Z, θ) to represent q . Let $(s(t), Z(t), \theta(t))$ be the solution of equation (2.8) initiated at (s^-, Z, θ) , and \tilde{t} be the time $(s(\tilde{t}), Z(\tilde{t}), \theta(\tilde{t}))$ hit Σ^+ . By definition $\mathcal{M}(q) = (s(\tilde{t}), Z(\tilde{t}), \theta(\tilde{t}))$. In what follows we write

$$\hat{Z} = Z(\tilde{t}), \quad \hat{\theta} = \theta(\tilde{t}).$$

Proposition 2.5. Denote $(\hat{Z}, \hat{\theta}) = \mathcal{M}(Z, \theta)$. We have

$$(2.15) \quad \begin{aligned} \hat{Z} &= P_L^+(\rho A_L + \phi_L(\theta)) + P_L Z + \mathcal{O}_{Z, \theta, p}(\mu) \\ \hat{\theta} &= \theta + \omega(L^+ + L^-) + \mathcal{O}_{Z, \theta, p}(\mu). \end{aligned}$$

Proof: The same as that of Proposition 3.2 of [WOk]. \square

D. The return map $\mathcal{F} = \mathcal{N} \circ \mathcal{M}$. We are now ready to compute the return map $\mathcal{F} = \mathcal{N} \circ \mathcal{M} : \Sigma^- \rightarrow \Sigma^-$. We use (\mathbb{X}, θ) to represent a point on Σ^- and denote $(\mathbb{X}_1, \theta_1) = \mathcal{F}(\mathbb{X}, \theta)$.

Proposition 2.6. The map $\mathcal{F} = \mathcal{N} \circ \mathcal{M} : \Sigma^- \rightarrow \Sigma^-$ is given by

$$(2.16) \quad \begin{aligned} \mathbb{X}_1 &= (\mu \varepsilon^{-1})^{\frac{\alpha}{\beta} - 1} [(1 + \mathcal{O}(\varepsilon)) P_L^+ \mathbb{F}(\mathbb{X}, \theta)]^{\frac{\alpha}{\beta}} \\ \theta_1 &= \theta + \omega(L^+ + L^-) + \frac{\omega}{\beta} \ln \mu^{-1} \varepsilon (1 + \mathcal{O}(\varepsilon)) P_L^+ - \frac{\omega}{\beta} \ln \mathbb{F}(\mathbb{X}, \theta) + \mathcal{O}_{\mathbb{X}, \theta, p}(\mu) \end{aligned}$$

where

$$(2.17) \quad \begin{aligned} \mathbb{F}(\mathbb{X}, \theta) &= (\rho A_L + \phi_L(\theta)) + P_L (P_L^+)^{-1} (1 + \mathcal{O}(\varepsilon)) \mathbb{X} \\ &\quad + (P_L^+)^{-1} (1 + P_L) \mathcal{O}_{\theta, p}(1) + \mathcal{O}_{\mathbb{X}, \theta, p}(\mu), \end{aligned}$$

and P_L, P_L^+ and $\phi_L(\theta)$ are as in (2.15) and (2.14).

Proof: By using Proposition 2.5 and Proposition 2.3(b), we have

$$\begin{aligned} \hat{Z} &= P_L (1 + \mathcal{O}(\varepsilon)) \mathbb{X} + P_L^+ (\rho A_L + \phi_L(\theta)) + P_L \mathcal{O}_{\theta, p}(1) + \mathcal{O}_{\mathbb{X}, \theta, p}(\mu) \\ \hat{\theta} &= \theta + \omega(L^+ + L^-) + \mathcal{O}_{\mathbb{X}, \theta, p}(\mu). \end{aligned}$$

Let $\hat{\mathbb{Y}}$ be the \mathbb{Y} -coordinate for $(\hat{Z}, \hat{\theta})$, we have from Proposition 2.3(a),

$$\hat{\mathbb{Y}} = (1 + \mathcal{O}(\varepsilon)) P_L^+ \mathbb{F}(\mathbb{X}, \theta)$$

where $\mathbb{F}(\mathbb{X}, \theta)$ is as in (2.17). We then obtain (2.16) by using (2.12). \square

3. PROOFS OF THEOREMS 1-4

Recall that $\ell = \ell_\lambda$ is the homoclinic solution of equation of Proposition 1.1 for equation (1.1), which we write as $\ell(s) = (a(s), b(s))$. $(u(s), v(s))$ is the unit tangent vector of ℓ at $\ell(s)$. Also recall that $\mathbb{H}_i(s)$, $i = 1, 2$ are as in (2.10) and $E(s)$ is as in (2.9). Let

$$(3.1) \quad \begin{aligned} A &= \int_{-\infty}^{\infty} \mathbb{H}_1(s) e^{-\int_0^s E(\tau) d\tau} ds \\ C(\omega) &= \int_{-\infty}^{\infty} \mathbb{H}_2(s) \cos(\omega s) e^{-\int_0^s E(\tau) d\tau} ds \\ S(\omega) &= \int_{-\infty}^{\infty} \mathbb{H}_2(s) \sin(\omega s) e^{-\int_0^s E(\tau) d\tau} ds. \end{aligned}$$

First we need to estimate $A, C(\omega)$ and $S(\omega)$.

Proposition 3.1. *There exists a $\lambda_0(R_\omega)$, sufficiently small depending on R_ω , such that for all $\omega \in (0, R_\omega)$ and all $\lambda \in (0, \lambda_0)$, we have*

- (i) $A > 1$; and
- (ii) $\sqrt{C^2(\omega) + S^2(\omega)} > \left(e^{-\frac{1}{2}\omega\pi} + e^{\frac{1}{2}\omega\pi} \right)^{-1}$.

Proposition 3.1 is proved in Section 4.

The return maps: To apply Proposition 2.6, we first fix a $R_\omega \gg 100\beta$. Then we pick a $\lambda \in (0, \lambda_0(R_\omega))$ satisfying the non-resonance assumption (H). We then chose $R_\rho, \varepsilon, R_\mu$ such that

$$R_\mu \gg \varepsilon^{-1} \gg R_\rho \gg \lambda^{-1}.$$

Recall that $p = (\omega, \rho, \ln \mu) \in \mathbb{P}$ represents the forcing parameter and

$$\Sigma^- = \{(\theta, \mathbb{X}) : \theta \in \mathbb{R}/(2\pi\mathbb{Z}), |\mathbb{X}| < 1\}.$$

Let $(\theta_1, \mathbb{X}_1) = \mathcal{F}(\theta, \mathbb{X})$ for $(\theta, \mathbb{X}) \in \Sigma^-$ where \mathcal{F} is from Proposition 2.6. We have

$$(3.2) \quad \begin{aligned} \theta_1 &= \theta + \mathbf{a} - \frac{\omega}{\beta} \ln \mathbb{F}(\theta, \mathbb{X}, p) \\ \mathbb{X}_1 &= \mathbf{b}[\mathbb{F}(\theta, \mathbb{X}, p)]^{\frac{\alpha}{\beta}} \end{aligned}$$

where

$$(3.3) \quad \begin{aligned} \mathbf{a} &= \frac{\omega}{\beta} \ln \mu^{-1} + \omega(L^+ + L^-) + \frac{\omega}{\beta} \ln(\varepsilon(1 + \mathcal{O}(\varepsilon))P_L^+ A_L) \\ \mathbf{b} &= (\mu\varepsilon^{-1})^{\frac{\alpha}{\beta}-1} [(1 + \mathcal{O}(\varepsilon))P_L^+ A_L]^{\frac{\alpha}{\beta}} \end{aligned}$$

and

$$(3.4) \quad \mathbb{F}(\theta, \mathbb{X}, p) = \rho + \mathbf{c} \sin \theta + \mathbf{k}\mathbb{X} + \mathbb{E}(\theta, p) + \mathcal{O}_{\theta, \mathbb{X}, p}(\mu),$$

in which

$$(3.5) \quad \begin{aligned} \mathbf{c} &= (A_L)^{-1} \sqrt{C_L^2 + S_L^2} \\ \mathbf{k} &= P_L (A_L P_L^+)^{-1} (1 + \mathcal{O}(\varepsilon)) \end{aligned}$$

and

$$(3.6) \quad \mathbb{E}(\theta, p) = (A_L P_L^+)^{-1} (1 + P_L) \mathcal{O}_{\theta, p}(1).$$

Note that in getting (3.2) we have changed $\theta + \omega L^- + c_0$ to θ where c_0 is such that $\tan c_0 = C_L^{-1} S_L$. $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{k}$ and $\mathbb{E}(\theta, p)$ are as follows:

(1) As parameters, \mathbf{b} and \mathbf{a} are functions of $p = (\omega, \rho, \ln \mu)$ and ε . If we regard ω, ρ and ε as been fixed, and $\mathcal{F}_\mu = \mathcal{F}$ as a one parameter family, then $\mathbf{b} \rightarrow 0$ as $\mu \rightarrow 0$. We can then think \mathcal{F} as an unfolding of the 1D maps

$$f(\theta) = \theta + \mathbf{a} - \frac{\omega}{\beta} \ln(\rho + \mathbf{c} \sin \theta + \mathbb{E}(\theta, 0))$$

where $\mathbf{a} \approx -\omega \beta^{-1} \ln \mu$. $\mathbf{a} \rightarrow +\infty$ as $\mu \rightarrow 0$.

(2) \mathbf{a} is a large number. But since it appears in the angular component we can module it by 2π . Varying μ from $R_\mu^{-1} > 0$ to zero is to run \mathbf{a} over $(\mathbf{a}_0, +\infty)$ for some $\mathbf{a}_0 \sim \omega \beta^{-1} \ln R_\mu$.

(3) \mathbf{c}, \mathbf{k} are independent of forcing parameters p but are functions of ε . From Proposition 3.1 and Lemma 2.1 we have

$$\mathbf{c} \neq 0, \quad \mathbf{k} \sim \varepsilon^{\frac{\alpha}{\beta}}$$

provided that ε is sufficiently small. Recall that the value of ε is selected after that of R_ω and R_ρ .

(4) From Lemma 2.1 and Proposition 3.1 we have

$$\mathbb{E}(\theta, \mu) \sim \varepsilon^{\beta \alpha^{-1}} \mathcal{O}_{\theta, p}(1).$$

When ε is sufficiently small, $\mathbb{E}(\theta, \mu)$ is a C^r -small perturbation to $\rho + \mathbf{c} \sin \theta$.

Proof of Theorem 1: To prove theorem 1, we fix the values of ω, μ and ε and regard $\rho \in (0, R_\rho)$ as the parameter in \mathcal{F} . Let

$$F(\theta, \rho) = \mathbb{F}(\theta, 0, p) = \rho + \mathbf{c} \sin \theta + \mathbb{E}(\theta, p) + \mathcal{O}_{\theta, p}(\mu)$$

and denote

$$M(\rho) = \min_{\theta \in S^1} F(\theta, \rho).$$

$M(\rho)$ is a continuous function of ρ .

We prove that $M(\rho)$ is monotonic in ρ . For this it suffices to prove that for all fixed $\theta \in S^1$,

$$\partial_\rho F(\theta, \rho) = 1 + \partial_\rho E(\theta, p) + \partial_\rho \mathcal{O}_{\theta, p}(\mu) > 1 - K \varepsilon^{\frac{\alpha}{\beta}} - K(\varepsilon) \mu > 0.$$

Next we prove that $W^s \cap W^u = \emptyset$ if and only if $M(\rho) > 0$. Let l_{loc}^u be the curve in Σ^- defined by $\mathbb{X} = 0$. From (3.2) we know that $\mathcal{F}^n(l_{loc}^u)$ is well-defined for all $n > 0$ if and only if $M(\rho) > 0$. We also know that $W^u \cap W^s = \emptyset$ if and only if $\mathcal{F}^n(l_{loc}^u)$ is well-defined for all $n > 0$ because W^u came out of l_{loc}^u .

To finish the proof of Theorem 1, it now suffices to observe that $M(R_\rho) > 0$ and $M(0) < 0$ because $\mathbf{c} \neq 0$. The surface S^* in \mathbb{P} is defined by $M(\rho) = 0$. In fact we have

$$(3.7) \quad S^*(\omega, \mu) = \mathbf{c} + \mathcal{O}(\varepsilon + \mu)$$

from $M(\rho) = 0$. □

Proof of Theorem 2: The formulas (3.2) for the return maps are identical to these obtained in Sect. 4.1 of [WOk], and Proposition 3.1 implies that assumption (H2) in Sect. 4.1 of [WOk] holds.² Assumption (H1) in Sect. 2.1 of [WOk] is (H). Consequently, Theorems 1-3 of [WOk] apply. A specific choice of the domain DH is defined in the paragraph entitled “specifications of parameters” of Sect. 4.1 of [WOk]. Theorem 2(i) is Theorem 1 of [WOk]; Theorem 2(ii) is Theorem 2 of [WOk]; and Theorem 2(iii) is Theorem 3 of [WOk]. □

Proof of Theorem 3: Let $\rho_0 > 0$ be such that

$$\rho_0 > 2A^{-1}\sqrt{C^2(0) + S^2(0)}$$

where $C(0), S(0)$ are the values of $C(\omega)$ and $S(\omega)$ at $\omega = 0$. For $(\omega, \rho, \mu) \in \mathbb{P}$ satisfying $\rho > \rho_0$, let

$$M(\omega, \rho, \mu) = \min_{\theta \in S^1} \mathbb{F}(\theta, 0, p).$$

We have

$$(3.8) \quad M(0, \rho, \mu) > \frac{1}{5}\rho_0 > 0.$$

We now define Q by letting

$$Q(\rho, \mu) = \min\{\omega > 0 : \quad \omega \leq 10^{-5}M(\omega, \rho, \mu)\mathbf{c}^{-1}(\omega); \quad M(\omega, \rho, \mu) \geq \frac{1}{10}\rho_0\}.$$

$Q(\rho, \mu)$ is well defined because of (3.8) and the fact that $M(\omega, \rho, \mu)$ is continuous in ω . In the rest of this proof we assume $(\omega, \rho, \mu) \in \mathbb{P}$ is such that $\omega < Q(\rho, \mu)$.

With (3.2) for \mathcal{F} and the way $Q(\rho, \mu)$ is defined above, our proof of Theorem 3 is the same as that of Theorem 1 of [WY4]. See Sect. 4.1 of [WY4]. For the existence of an attracting invariant curve, we observe that $D\mathcal{F}$ has a dominating splitting on Σ^- (Sect. 4.1.1 of [WY4]). Theorem 3(2) then follows from Denjoy theory and a direct application of a Theorem of Herman [He] to the circle diffeomorphisms induced on the attracting invariant curves (Sect. 4.1.2 of [WY4]).

To be more precise, we let $\mathbf{v} = (u, v)$ be a tangent vector of Σ^- at q and let $s(\mathbf{v}) = vu^{-1}$ for $q = (\theta, \mathbb{X}) \in \Sigma^-$. $s(\mathbf{v})$ is the slope of \mathbf{v} . Let $\mathcal{C}_h(q)$ be the collection of all \mathbf{v} satisfying $|s(\mathbf{v})| < \frac{1}{100}$, and $\mathcal{C}_v(q)$ be the collection of all \mathbf{v} satisfying $|s(\mathbf{v})| > 100$. We first prove that

- (i) on Σ^- , we have $D\mathcal{F}_q(\mathcal{C}_h(q)) \subset \mathcal{C}_h(\mathcal{F}(q))$, and $|D\mathcal{F}_q(\mathbf{v})| > \frac{1}{2}|\mathbf{v}|$;
- (ii) on $\mathcal{F}(\Sigma^-)$, we have $D\mathcal{F}_q^{-1}(\mathcal{C}_v(q)) \subset \mathcal{C}_v(\mathcal{F}(q))$ and $|D\mathcal{F}_q^{-1}(\mathbf{v})| > 100|\mathbf{v}|$ for all $\mathbf{v} \in \mathcal{C}_h(q)$

²Though there is a difference in writing: in this paper we kept a factor ρ inside of \mathbb{F} while in [WOk] we gave it to \mathbf{b} and \mathbf{a} . With the current writing the proof of Theorem 1 is easier to present.

To prove (i) we compute $D\mathcal{F}$ by using (3.2). Let $(\theta_1, \mathbb{X}_1) = \mathcal{F}(\theta, \mathbb{X})$, we have

$$(3.9) \quad D\mathcal{F} = \begin{pmatrix} \frac{\partial \theta_1}{\partial \theta} & \frac{\partial \theta_1}{\partial \mathbb{X}} \\ \frac{\partial \mathbb{X}_1}{\partial \theta} & \frac{\partial \mathbb{X}_1}{\partial \mathbb{X}} \end{pmatrix} = \begin{pmatrix} 1 - \omega \beta^{-1} \frac{1}{\mathbb{F}} \frac{\partial \mathbb{F}}{\partial \theta} & \omega \beta^{-1} \frac{1}{\mathbb{F}} \frac{\partial \mathbb{F}}{\partial \mathbb{X}} \\ \alpha \beta^{-1} \mathbf{b} \mathbb{F}^{\alpha \beta^{-1} - 1} \frac{\partial \mathbb{F}}{\partial \theta} & \alpha \beta^{-1} \mathbf{b} \mathbb{F}^{\alpha \beta^{-1} - 1} \frac{\partial \mathbb{F}}{\partial \mathbb{X}} \end{pmatrix}$$

where $\mathbb{F} = \mathbb{F}(\theta, \mathbb{X}, p)$ is as in (3.4) and

$$\begin{aligned} \frac{\partial \mathbb{F}}{\partial \theta} &= \mathbf{c} \cos \theta + \varepsilon^{\beta \alpha^{-1}} \mathcal{O}_{\theta, p}(1) + \mathcal{O}_{\theta, \mathbb{X}, \mathbf{a}}(\mu) \\ \frac{\partial \mathbb{F}}{\partial \mathbb{X}} &= \mathbf{k} + \mathcal{O}_{\theta, \mathbb{X}, p}(\mu). \end{aligned}$$

(i) then follows from $\mathbf{b} \approx \mu^{\lambda \beta^{-1}} \ll 1$ and

$$(3.10) \quad \left| \omega \beta^{-1} \frac{1}{\mathbb{F}} \frac{\partial \mathbb{F}}{\partial \theta} \right| < 10^{-4}$$

where (3.10) is from $\omega < Q(\rho, \mu)$. To prove (ii) we use again (3.10) and

$$(3.11) \quad D\mathcal{F}^{-1} = \frac{1}{\alpha \beta^{-1} \mathbf{b} \mathbb{F}^{\alpha \beta^{-1} - 1} \frac{\partial \mathbb{F}}{\partial \mathbb{X}}} \begin{pmatrix} \alpha \beta^{-1} \mathbf{b} \mathbb{F}^{\alpha \beta^{-1} - 1} \frac{\partial \mathbb{F}}{\partial \mathbb{X}} & -\omega \beta^{-1} \frac{1}{\mathbb{F}} \frac{\partial \mathbb{F}}{\partial \mathbb{X}} \\ -\alpha \beta^{-1} \mathbf{b} \mathbb{F}^{\alpha \beta^{-1} - 1} \frac{\partial \mathbb{F}}{\partial \theta} & 1 - \omega \beta^{-1} \frac{1}{\mathbb{F}} \frac{\partial \mathbb{F}}{\partial \theta} \end{pmatrix}.$$

With both (i) and (ii), Theorem 3(1) follows now from the standard graph transformation argument. Proof for Theorem 3(2) is the same as the proof of Theorem 1(a)(b) of [WY4]. See Sect. 4.1.2 of [WY4], where T is the parameter \mathbf{a} for here. \square

Proof of Theorem 4: Theorem 4(2) is proved by applying the theory of rank one maps of [WY1] and [WY2] to \mathcal{F} of (3.2). The domain DC is such that

$$\frac{202}{99} \frac{\sqrt{C^2(\omega) + S^2(\omega)}}{A} < \rho < \frac{396}{101} \frac{\sqrt{C^2(\omega) + S^2(\omega)}}{A}.$$

We also need to assume that ω is sufficiently large for $(\omega, \rho) \in DC$ in order to prove that \mathcal{F}_μ is an admissible family. For more details see Section 6 of [WO].

Theorem 4(1) is a specific case of Theorem 3 of [LW]. \square

4. PROOF OF PROPOSITION 3.1

In this section we prove Proposition 3.1. We start with the conservative part of the autonomous equation (1.1), that is, the equation

$$(4.1) \quad \frac{d^2 q}{dt^2} - q + q^3 = 0.$$

Denote $p = \frac{dq}{dt}$. We write equation (4.1) as

$$(4.2) \quad \frac{dq}{dt} = p, \quad \frac{dp}{dt} = q - q^3.$$

With a simple linear change of coordinates

$$x = \frac{1}{2}(q - p), \quad y = \frac{1}{2}(q + p),$$

we write equation (4.1) in (x, y) as

$$(4.3) \quad \frac{dx}{dt} = -x + \frac{1}{2}(x+y)^3, \quad \frac{dy}{dt} = y - \frac{1}{2}(x+y)^3.$$

Let

$$a(t) = \frac{2\sqrt{2}e^{3t}}{(1+e^{2t})^2}, \quad b(t) = \frac{2\sqrt{2}e^t}{(1+e^{2t})^2}.$$

$(x, y) = (0, 0)$ is a homoclinic saddle and $(x, y) = (a(t), b(t))$ is a homoclinic solution of equation (4.3) initiated at $(\sqrt{2}, \sqrt{2})$. Denote

$$\ell = \{\ell(t) = (a(t), b(t)), \ t \in \mathbb{R}\}.$$

It follows that

$$(4.4) \quad \begin{aligned} u(t) &= \frac{-(e^{2t} - 3)}{\sqrt{(e^{2t} - 3)^2 + (e^{-2t} - 3)^2}} \\ v(t) &= \frac{e^{-2t} - 3}{\sqrt{(e^{2t} - 3)^2 + (e^{-2t} - 3)^2}}, \end{aligned}$$

where

$$(u(t), v(t)) = \left| \frac{d}{dt} \ell(t) \right|^{-1} \frac{d}{dt} \ell(t)$$

is the unit tangent vector of ℓ at $\ell(t)$.

Let us re-write equation (4.3) as

$$(4.5) \quad \frac{dx}{dt} = -x + f(x, y) \quad \frac{dy}{dt} = y + g(x, y)$$

where

$$f(x, y) = \frac{1}{2}(x+y)^3, \quad g(x, y) = -\frac{1}{2}(x+y)^3.$$

Let

$$\begin{aligned} E(t) &= v^2(t)(-1 + \partial_x f(a(t), b(t))) + u^2(t)(1 + \partial_y g(a(t), b(t))) \\ &\quad - u(t)v(t)(\partial_y f(a(t), b(t)) + \partial_x g(a(t), b(t))). \end{aligned}$$

We have

$$(4.6) \quad E(t) = -\frac{(e^{-2t} - 3)^2 - (e^{2t} - 3)^2}{(e^{2t} - 3)^2 + (e^{-2t} - 3)^2} \left(1 - \frac{12e^{2t}}{(1+e^{2t})^2} \right).$$

Let

$$(4.7) \quad K(s) = -\int_0^s E(s) ds.$$

Lemma 4.1. For $s \in (-\infty, \infty)$,

$$K(s) = \frac{1}{2} \ln \frac{8e^{2s}((1 - 3e^{2s})^2 + e^{4s}(e^{2s} - 3)^2)}{(e^{2s} + 1)^6}.$$

Proof: Let $x = e^{2t}$. We have³

$$K(s) = \frac{1}{2} \int_1^{e^{2s}} \frac{1}{x} \frac{(1-3x)^2 - x^2(x-3)^2}{(1-3x)^2 + x^2(x-3)^2} \left(1 - \frac{12x}{(1+x)^2}\right) dx.$$

Observe that

$$(1-3x)^2 + x^2(x-3)^2 = ((x-a)^2 + a^2)((x-b)^2 + b^2)$$

where $a, b > 0$ satisfying $a + b = 3$, $ab = \frac{1}{2}$. We have

$$\begin{aligned} K(s) &= \frac{1}{2} \int_1^{e^{2s}} \frac{(-x^4 + 6x^3 - 6x + 1)(x^2 - 10x + 1)}{x(x+1)^2((x-a)^2 + a^2)((x-b)^2 + b^2)} dx \\ &= \frac{1}{2} \int_1^{e^{2s}} \left(\frac{1}{x} - \frac{6}{1+x} + \frac{4x^3 - 18x^2 + 36x - 6}{((x-a)^2 + a^2)((x-b)^2 + b^2)} \right) dx \\ &= \frac{1}{2} \int_1^{e^{2s}} \left(\frac{1}{x} - \frac{6}{1+x} + \frac{2(x-a)}{(x-a)^2 + a^2} + \frac{2(x-b)}{(x-b)^2 + b^2} \right) dx. \end{aligned}$$

We then have

$$\begin{aligned} K(s) &= \frac{1}{2} \ln \frac{8e^{2s}((e^{2s}-a)^2 + a^2)((e^{2s}-b)^2 + b^2)}{(e^{2s}+1)^6} \\ &= \frac{1}{2} \ln \frac{8e^{2s}((1-3e^{2s})^2 + e^{4s}(e^{2s}-3)^2)}{(e^{2s}+1)^6}. \end{aligned}$$

□

Through direct evaluation using Lemma 4.1, we obtain the values of the following definite integrals as

$$(4.8) \quad \int_{-\infty}^{\infty} (u(s) + v(s))(b(s) + (a(s))^2(b(s) - a(s)))e^{-\int_0^s E(t)dt} ds = \frac{16}{15}.$$

We have two more integrals to evaluate, and they are

$$(4.9) \quad \begin{aligned} C(\omega) &= \int_{-\infty}^{\infty} (u(s) + v(s))(b(s) + a(s))^2 \cos(\omega s) e^{-\int_0^s E(t)dt} ds, \\ S(\omega) &= \int_{-\infty}^{\infty} (u(s) + v(s))(b(s) + a(s))^2 \sin(\omega s) e^{-\int_0^s E(t)dt} ds. \end{aligned}$$

Lemma 4.2. *We have*

$$C(\omega) = \frac{16\sqrt{2}\pi}{e^{-\frac{1}{2}\omega\pi} + e^{\frac{1}{2}\omega\pi}}, \quad S(\omega) = -\frac{2\sqrt{2}}{3} \cdot \frac{\omega(1+\omega^2)}{e^{-\frac{1}{2}\omega\pi} + e^{\frac{1}{2}\omega\pi}}.$$

Proof: Using Lemma 4.1, we have

$$C(\omega) = 16\sqrt{2}I_c, \quad S(\omega) = 16\sqrt{2}I_s$$

³The author would like to thank Ali Oksasoglu for making this computation.

where

$$I_c = \int_{-\infty}^{\infty} \frac{e^{3s}(1 - e^{2s})}{(e^{2s} + 1)^4} \cdot \cos(\omega s) \cdot ds$$

$$I_s = \int_{-\infty}^{\infty} \frac{e^{3s}(1 - e^{2s})}{(e^{2s} + 1)^4} \cdot \sin(\omega s) \cdot ds.$$

Let $I = I_c + iI_s$.

$$I = \int_{-\infty}^{\infty} \frac{e^{3s}(1 - e^{2s})}{(e^{2s} + 1)^4} \cdot e^{i\omega s} \cdot ds.$$

We evaluate I as follows. On the $z = x + iy$ plan, let

$$\ell_1 = \{x, \quad x : -\infty \rightarrow \infty\}, \quad \ell_2 = \{x + \pi i, \quad x : \infty \rightarrow -\infty\}$$

and

$$f(z) = \frac{e^{3z}(1 - e^{2z})}{(e^{2z} + 1)^4} \cdot e^{i\omega z}.$$

We have

$$\int_{\ell_1} f(z) dz = I, \quad \int_{\ell_2} f(z) dz = e^{-\omega\pi} I.$$

So by the residue theorem,

$$(1 + e^{-\omega\pi})I = 2\pi i \operatorname{Res}(f(z))_{z=\frac{\pi i}{2}}.$$

Let $t = z - \frac{\pi i}{2}$, we write $f(z)$ as

$$\begin{aligned} f(z) &= -ie^{-\frac{1}{2}\omega\pi} \frac{e^{3t}(e^{2t} + 1)}{(1 - e^{2t})^4} \cdot e^{i\omega t} \\ &= -\frac{i}{16} e^{-\frac{1}{2}\omega\pi} \frac{e^{(5+\omega i)t} + e^{(3+\omega i)t}}{t^4(1 + t + \frac{2}{3}t^2 + \frac{1}{3}t^3 + \mathcal{O}(t^4))}. \end{aligned}$$

We have

$$\begin{aligned} e^{(5+i\omega)t} + e^{(3+i\omega)t} &= 1 + (8 + 2i\omega)t + (17 + 8i\omega - \omega^2)t^2 \\ &\quad + \frac{1}{3}(76 - 12\omega^2 + (51\omega - \omega^3)i)t^3 + \mathcal{O}(t^4). \end{aligned}$$

We also have

$$\frac{1}{(1+x)^4} = 1 - 4x + 10x^2 - 20x^3 + \dots$$

so

$$(1 + t + \frac{2}{3}t^2 + \frac{1}{3}t^3)^{-4} = 1 - 4t + \frac{22}{3}t^2 - 8t^3 + \mathcal{O}(t^4).$$

Consequently,

$$\operatorname{Res}(f(z)) = -\frac{i}{16} e^{-\frac{1}{2}\omega\pi} (8 - \frac{1}{3}\omega(1 + \omega^2)i),$$

from which it follows that

$$I = \frac{1}{8} \frac{\pi e^{-\frac{1}{2}\omega\pi}}{1 + e^{-\omega\pi}} (8 - \frac{1}{3}\omega(1 + \omega^2)i).$$

In conclusion, we have

$$C(\omega) = \frac{16\sqrt{2}\pi}{e^{-\frac{1}{2}\omega\pi} + e^{\frac{1}{2}\omega\pi}}, \quad S(\omega) = -\frac{2\sqrt{2}}{3} \cdot \frac{\omega(1+\omega^2)}{e^{-\frac{1}{2}\omega\pi} + e^{\frac{1}{2}\omega\pi}}.$$

□

Proof of Proposition 3.1 We use subscript λ to stress the role of λ . We denote equation (2.4) as $(2.4)_\lambda$, and the homoclinic solution ℓ_λ of Proposition 1.1 as

$$x = a_\lambda(t), \quad y = b_\lambda(t).$$

Similarly, E_λ is defined through (2.9), and A_λ , $C_\lambda(\omega)$ and $S_\lambda(\omega)$ are defined through (3.1). We need to estimate A_λ and $C_\lambda(\omega)$.

In this section we have so far computed A_λ and C_λ, S_λ for $\lambda = 0$. We have from (4.8) and Lemma 4.2

$$(4.10) \quad A_0 = \frac{16}{15}, \quad C_0(\omega) = \frac{16\sqrt{2}\pi}{e^{-\frac{1}{2}\omega\pi} + e^{\frac{1}{2}\omega\pi}}.$$

We caution that $\alpha_0 = \beta_0 = 1$ so for $\lambda = 0$ the saddle point $(x, y) = (0, 0)$ is not dissipative. However, since $-\alpha + \beta = -\lambda$ from (1.3), $(x, y) = (0, 0)$ is a dissipative saddle for $\lambda > 0$. We estimate A_λ and $C_\lambda(\omega)$ through $A_0, C_0(\omega)$.

Let U_ε be the ε -neighborhood of $(0, 0)$ in the (x, y) -plane, and $s_\lambda^+(\varepsilon)$ be the time ℓ_λ enters U_ε . Similarly let $-s_\lambda^-(\varepsilon)$ be the time ℓ_λ exits U_ε . Then there exists $\varepsilon_0 > 0$ and K independent of λ such that for all $\lambda > 0$ sufficiently small, and all $s \in (s_\lambda^+(\varepsilon_0), +\infty)$,

$$(4.11) \quad |u_\lambda(s)| > 1 - K\varepsilon_0, \quad |v_\lambda(s)| < K\varepsilon_0.$$

Similarly we have, for all $s \in (-\infty, -s_\lambda^-(\varepsilon_0))$,

$$(4.12) \quad |v_\lambda(s)| > 1 - K\varepsilon_0, \quad |u_\lambda(s)| < K\varepsilon_0.$$

Both (4.11) and (4.12) follow directly from the fact that the local stable and the unstable manifold of equation (1.2) are tangent to the x and the y -axis respectively at $(x, y) = (0, 0)$, and their curvatures are bounded uniformly in λ . From (4.11) and (4.12) we have, for all $s \in (-\infty, -s_\lambda^-(\varepsilon_0)) \cup (s_\lambda^+(\varepsilon_0), +\infty)$,

$$(4.13) \quad |E_\lambda(s)| > \frac{1}{2}$$

provided that λ are sufficiently small. Let

$$A_{\lambda, s^-, s^+} = \int_{-s^-}^{s^+} (u_\lambda(s) + v_\lambda(s))(b_\lambda(s) + a_\lambda(s))^2(b_\lambda(s) - a_\lambda(s))e^{-\int_0^s E_\lambda(t)dt} ds.$$

We have from (4.13) that

$$(4.14) \quad |A_\lambda - A_{\lambda, s^-, s^+}| < \frac{1}{100}$$

for all λ that is sufficiently small provided that $s^- > s_\lambda^-(\varepsilon_0), s^+ > s_\lambda^+(\varepsilon_0)$.

Let $s^- = s_0^-(\frac{1}{2}\varepsilon_0)$, $s^+ = s_0^+(\frac{1}{2}\varepsilon_0)$. From Proposition 1.1(ii), we have $\ell_\lambda(-s^-), \ell_\lambda(s^+) \in U_{\varepsilon_0}$ provided that $\lambda \in (0, K^{-1}(\varepsilon_0))$ where $K(\varepsilon_0) > 0$ is a dependent of ε_0 . We also have $\ell_\lambda(s) \cap U_{\frac{1}{4}\varepsilon_0} = \emptyset$ for all $s \in (-s^-, s^+)$, and this implies that for all $s \in (-s^-, s^+)$

$$(4.15) \quad |u_\lambda(s) - u_0(s)|, \quad |v_\lambda(s) - v_0(s)| < K(\varepsilon_0)\lambda.$$

(4.15) and Proposition 1.1(ii) together implies

$$(4.16) \quad |A_{\lambda, s^-, s^+} - A_{0, s^-, s^+}| < K(\varepsilon_0)\lambda.$$

That $A_\lambda > 1$ follows now from (4.16), (4.14) and $A_0 > \frac{16}{15}$.

Estimates on $C_\lambda(\omega)$ is similar. Because $C_\lambda(\omega)$ is a function in ω , so would be the constant $K(\varepsilon_0)$ in (4.16). This is why λ_0 in this proposition is a dependent of R_ω . \square

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